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Mild Parameterization and the Rational Points of a Pfaff Curve

by

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Abstract. This paper is concerned with the density of rational points on the graph of a non-algebraic pfaffian function. It presents some conjectures and partial results.

1. Introduction

This paper is concerned with the density of rational points on a pfaff curve. I begin with the definitions required to state the result and ambient conjectures.

1.1. DEFINITION. Let $H : \mathbb{Q} \to \mathbb{R}$ be the usual height function, $H(a/b) = \max(|a|, b)$ for $a, b \in \mathbb{Z}$ with $b > 0$ and $(a, b) = 1$. Define $H : \mathbb{Q}^n \to \mathbb{R}$ by $H(\alpha_1, \alpha_2, \ldots, \alpha_n) = \max_{1 \le j \le n}(H(\alpha_j))$. For a set $X \subset \mathbb{R}^n$ define $X(\mathbb{Q}) = X \cap \mathbb{Q}^n$ and, for $H \ge 1$, put

$$X(\mathbb{Q}, H) = \{P \in X(\mathbb{Q}) : H(P) \le H\}.$$

The *density function* of X is the function

$$N(X, H) = \#X(\mathbb{Q}, H).$$

1.2. DEFINITION ([4, 2.1]). Let $U \subset \mathbb{R}^n$ be an open domain. A *pfaffian chain* of order $r \ge 0$ and degree $\alpha \ge 1$ in U is a sequence of real analytic functions $f_1, \ldots, f_r$ in U satisfying differential equations

$$df_j = \sum_{i=1}^{n} g_{ij}(\mathbf{x}, f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_j(\mathbf{x}))dx_i$$

for $j = 1, \ldots, r$, where $\mathbf{x} = (x_1, \ldots, x_n)$ and $g_{ij} \in \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_r]$ of degree $\le \alpha$. A function f on U is called a *pfaffian function* of order r and degree (α, β) if $f(\mathbf{x}) = P(\mathbf{x}, f_1(\mathbf{x}), \ldots, f_r(\mathbf{x}))$, where P is a polynomial of degree at most $\beta \ge 1$. In this paper always $n = 1$, so $\mathbf{x} = x$. A *pfaff curve* X is the graph of a pfaffian function f on some connected subset of its domain. The *order* and *degree* of X will be taken to be the order and degree of f.

The usual elementary functions e^x, $\log x$ (but not $\sin x$ on all $\mathbb{R}$), algebraic functions, and sums, products and compositions of these are pfaffian functions, such as e.g. $e^{-1/x}$, e^{e^x}, etc; see [8, 4]. Note that for non-semi-algebraic X, the set $X(\mathbb{Q})$ may be infinite (e.g. 2^x), or of unknown size (e.g. e^{e^x}).

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Suppose X is a pfaff curve that is not semialgebraic. Since the *structure* generated by pfaffian functions is *o-minimal* (for these terms see [3, 14]), an estimate of the form

$$N(X, H) \le c(X, \varepsilon) H^{\varepsilon}$$

for all positive ε (and, with suitable hypotheses, in all dimensions) follows from [12].

I have shown in [11] that there is an explicit function $c(r, \alpha, \beta)$ with the following property. Suppose X is a nonalgebraic pfaff curve of order r and degree (α, β). Let $H \ge c(r, \alpha, \beta)$. Then

$$N(X, H) \le \exp(5\sqrt{\log H}) .$$

As noted in [12], no such quantification of the bound $c(X, \varepsilon)H^{\varepsilon}$ can hold in general, e.g. for bounded subanalytic sets.

The following is an extrapolation of part of the one-dimensional case of conjecture [12, 1.11] (the latter refers to sets in the structure generated by e^x.)

1.3. CONJECTURE. Let $X \subset \mathbb{R}^2$ be a pfaff curve, and suppose that X is not semialgebraic. There are constants $\beta, \gamma > 0$ such that (for $H \ge e$)

$$N(X, H) \le \beta(\log H)^{\gamma} .$$

1.4. DEFINITION. Let $f : (0, 1) \to [-1, 1]$ be a smooth function, and $B, C > 0$. We will write $f \in M(B, C)$ if

$$\frac{|f^{(k)}(x)|}{k!} \le (Bk^C)^k$$

for all $k \in \mathbb{N} = \{0, 1, 2, \dots\}$ (where $0^0 = 1$; so the condition is always satisfied for $k = 0$). We call f *mild* if there are constants B, C such that $f \in M(B, C)$. A set $X \subset \mathbb{R}^2$ will be said to admit *mild parameterization* if there is a finite collection Φ of maps $\phi : (0, 1) \to \mathbb{R}^2$ whose coordinate functions are mild, and such that the union of images is X.

1.5. THEOREM. *Let $X \subset [-1, 1]^2$ be a pfaff curve, and suppose that X admits mild parameterization. Then there are constants $\beta, \gamma > 0$ such that* (*for $H \ge e$*)

$$N(X, H) \le \beta(\log H)^{\gamma} .$$

In particular, the theorem holds for a pfaff curve $X : y = f(x), x \in (a, b) \subset (0, 1)$ when all functions in the chain for f are bounded on (a, b), since in that case f is analytic on a neighbourhood of $[a, b]$, and hence belongs to some $M(B, 0)$ (see the discussion near 2.2 below). And by this remark, for any such $X : y = f(x)$, it holds for the restriction of f to any subinterval $(a + \delta, b - \delta)$, with constants $\beta_\delta, \gamma_\delta$. The theorem also holds for any pfaff curve that is subanalytic — by the Uniformization theorem ([1, 0.1]). The hypotheses may indeed hold for *any* pfaff curve.

1.6. CONJECTURE. Let $X \subset [-1, 1]^2$ be a pfaff curve. Then X admits mild parameterization.

Some token evidence for this conjecture is presented in §4. Since the transformations $x \mapsto \pm 1/x$ preserve height and the class of pfaffian function, Conjecture 1.3 follows from its restriction to curves in $[-1,1]^2$. Thus, by Theorem 1.5, Conjecture 1.6 implies Conjecture 1.3.

The idea of using a reparameterization of a set $X \subset \mathbb{R}^2$ in order to realize it as the union of images of functions whose derivatives up to some prescribed order are bounded is exploited for diophantine purposes in [12]. The possibility of doing this in such a way as to require a bounded number of maps, for all sets in a parameterized family, follows from a suitable adaptation of results of Yomdin and Gromov [5, 15] in the semialgebraic setting.

The observation exploited in the present paper is that a quite weak control of derivatives to *all* orders enables strong control of the rational points.

2. Mild maps

The basic maps to be considered are smooth (i.e. C^∞) maps $(0,1) \to [-1,1]$. Denote this class of maps by $\mathcal{C}$.

Let $f \in \mathcal{C}$. Set $A_0(f) = 1$ and for $k = 1, 2, \ldots$ define

$$A_k(f) = \max\left(1, \sup\left\{\left|\frac{f^{(\kappa)}(x)}{\kappa!}\right|^{1/\kappa} : 1 \le \kappa \le k, x \in (0,1)\right\}\right)$$

(with $A_k(f) = \infty$ if some $f^{(\kappa)}$, $\kappa \le k$ is unbounded on $(0,1)$). Thus, for any $x \in (0,1)$ and $\kappa, k \in \mathbb{N} = \{0, 1, 2, \ldots\}$ with $\kappa \le k$,

$$\frac{|f^{(\kappa)}(x)|}{\kappa!} \le A_k(f)^k.$$

2.1. Proposition. *Let $f, g \in \mathcal{C}$ and suppose p, q, k are non-negative integers. Then*

$$A_k(f^p g^q) \le (k+1)^{(p+q)/k} \max(A_k(f), A_k(g)).$$

Proof. Expand

$$(f^p g^q)^{(k)} = \sum_{\substack{i_1+i_2+\cdots+i_p+\\ j_1+j_2+\cdots+j_q=k}} \frac{k!}{i_1!i_2!\cdots i_p!j_1!j_2!\cdots j_q!} f^{(i_1)} f^{(i_2)} \ldots f^{(i_p)} g^{(j_1)} g^{(j_2)} \ldots g^{(j_q)}$$

whence, for any $t \in (0,1)$,

$$\begin{aligned}\frac{|(f^p g^q)^{(k)}(t)|}{k!} &\le (k+1)^{p+q} A_k(f)^{i_1+\cdots+i_p} A_k(g)^{j_1+\cdots+j_q}\\ &\le (k+1)^{p+q} \max(A_k(f), A_k(g))^k\end{aligned}$$

and the proposition follows. □

If $\phi : (0,1) \to [-1,1]^2$ with coordinate functions $f, g \in \mathcal{C}$ and $D \in \mathbb{N}$, set

$$\alpha_D(\phi) = \left(\prod_{k=0}^{D-1} \max\big(A_k(f), A_k(g)\big)^k\right)^{\frac{2}{D(D-1)}}.$$

If the function $f \in \mathcal{C}$ has the property that $A_k(f) \le B < \infty$, so that $f \in M(B, 0)$, then ($B \ge 1$ and) the Taylor series converges in an interval of radius b for any $b < B$ and any $x \in (0, 1)$. This means that f is analytic on an open neighbourhood of $[0, 1]$. Conversely, if f is analytic on an open neighbourhood of $[0, 1]$ then the Taylor series converges on a disk of radius $b > 0$ about every $x \in (0, 1)$, whence $f \in M(B, 0)$ for some B (with $B \ge 1$). If $\phi : (0, 1) \to [-1, 1]^2$ with $\phi(x) = (x, f(x))$ then $\alpha_D(\phi) \le B$ for all D.

For $\phi : (0, 1) \to \mathbb{R}^2$ we will write $\phi \in M(B, C)$ if $\phi((0, 1)) \subset [-1, 1]^2$ and the coordinate functions f, g of ϕ both belong to $M(B, C)$.

2.2. PROPOSITION. *Let $\phi \in M(B, C)$. Then $\alpha_D(\phi) \le BD^C$.*

Proof. Observe

$$\log \alpha_D(\phi) \le \frac{2}{D(D-1)} \sum_{k=1}^{D-1} (k \log B + Ck \log k) .$$

Now

$$\sum_{k=1}^{D-1} k \log k \le \int_1^D x \log x \, dx = \left[\frac{x^2 \log x}{2} - \frac{x^2}{4} \right]_1^D$$
$$\le \left(\frac{D(D-1) \log D}{2} + \frac{D \log D}{2} - \frac{D^2}{4} + \frac{1}{4} \right)$$

and the sum of the last 3 terms is (for $D \ge 1$) non-positive. $\square$

Let M be a set of monomials in x, y. If $m = x^i y^j \in M$ and $f(t), g(t)$ are functions, write $m(f, g)$ for the function $f(t)^i g(t)^j$.

2.3. PROPOSITION. *Let $d \ge 1$. Let $M = \{m_j\}$ denote the set of monomials of degree $\le d$ and set $D = (d+1)(d+2)/2 = \#M$. Let $f, g \in C$ with bounded derivatives up to order D. Let $0 < t_1 < t_2 < \cdots < t_D < 1$ and put*

$$\Delta = \det(m_j(f, g)(t_i)) .$$

Then

$$|\Delta| \le V \, D! \, D^{2dD/3} \, \alpha_D(\phi)^{D(D-1)/2} ,$$

where $V = V(t_1, t_2, \ldots, t_D)$ is the Vandermonde determinant.

Proof. For any functions ϕ_j possessing $D - 1$ continuous derivatives on an interval containing the points,

$$\det(\phi_j(t_i)) = V \det L$$

where V is the Vandermonde and

$$L = \left(\frac{\phi_j^{(i-1)}(\xi_{ij})}{(i-1)!} \right)$$

for suitable intermediate points ξ_{ij} (see [2]).

If $m \in M$ is a monomial of degree δ then $A_k(m(f, g)) \le (k+1)^{\delta/k} \max(A_k(f), A_k(g))$ by 2.1 and so

$$\frac{|m^{(k)}(\xi)|}{k!} \le (k+1)^{\delta} \max(A_k(f), A_k(g))^k .$$

Therefore

$$|\Delta| \le V\, D! \prod_{m\in M} D^{\deg m} \prod_{i=1}^{D} \max(A_{i-1}(f), A_{i-1}(g))^{i-1} .$$

The sum of exponents $\deg m$ is the total degree of x, y in all the monomials in M, which is 2/3 of the total degree of the corresponding homogeneous monomials in x, y, z of degree d, and this sum is dD. The conclusion follows. □

2.4. PROPOSITION. *Let $d \ge 5$. Let $\phi \in M(B, C)$ with image X. Let $H \ge 1$. Then $X(\mathbb{Q}, H)$ is contained in the union of at most*

$$6BD^C H^{8/(d+3)}$$

algebraic curves of degree d.

Proof. Let $I = (a, b)$ be a subinterval of (0, 1). Suppose that the points of $X(\mathbb{Q}, H)$ corresponding to $t \in I$ do *not* all lie on a curve of degree d. It follows that there exists $a < t_1 < \cdots < t_D < b$ such that $f(t_i), g(t_i) \in \mathbb{Q}$ with $H(f(t_i), g(t_i)) \le H$ for all i and

$$0 \ne \Delta = \det(m_j(f(t_i), g(t_i))) .$$

The height condition on the points $(f(t_i), g(t_i))$ controls the size of the denominators of the entries, giving

$$H^{2dD}|\Delta| \ge 1 .$$

On the other hand,

$$|\Delta| \le |I|^{D(D-1)/2}\, D!\, e^{2dD/3}\, \alpha_D(\phi)^{D(D-1)/2}$$

by Proposition 2.3. (In fact the Vandermonde can be estimated a bit better using the transfinite diameter, saving an additional factor of 4.) It follows that

$$|I| \ge (H^{2dD} D!\, e^{2dD/3} \alpha_D(\phi)^{D(D-1)/2})^{-2/(D(D-1))} .$$

Therefore, if I is an interval with $|I|$ less than the righthand side above, then the corresponding points of $X(\mathbb{Q}, H)$ all lie on a curve of degree d. The interval (0, 1) may be covered by

$$1 + (H^{2dD}\, D!\, D^{2dD/3})^{2/(D(D-1))} \alpha_D(\phi)$$

such intervals. Now $(D!)^{2/(D(D-1))} \le e^{0.3611}$ for $d \ge 5$, while $(D^{2dD/3})^{2/(D(D-1))} = D^{8/(3(d+3))} \le 3$ and so

$$\le 1 + 3e^{1.3611} \alpha_D(\phi) H^{8/(d+3)} \le 6\alpha_D(\phi) H^{8/(d+3)}$$

intervals (and hence curves) suffice. □

3. Pfaff curves; proof of 1.5

3.1. PROPOSITION. *Let f be a pfaffian function of order $r \geq 1$ and degree (α, β), and suppose f is not algebraic. Suppose $P(x, y) \in \mathbb{R}[x, y]$ of degree d. Then the equation*

$$P(x, f(x)) = 0$$

has at most

$$2^{r(r-1)/2+1} d\beta(\alpha + d\beta)^r$$

real solutions.

Proof. Observe that $P(x, f(x))$ is a pfaffian function of order $r \geq 1$ and degree $(\alpha, d\beta)$. Since f is not algebraic, all solutions to $P(x, f(x)) = 0$ are isolated. The conclusion follows from [4, 3.3], which asserts the exhibited bound for the number of connected components of the zero-set of such a function. □

3.2. PROOF OF 1.5. Let Φ be a collection of maps $\phi : (0, 1) \to \mathbb{R}^2$ giving a mild parameterization of X, and suppose $\phi \in M(B, C)$ for all $\phi \in \Phi$. Let $H \geq e$ and set $d = [\log H]$, where $[t]$ denotes the largest integer $\leq t$. Then $D = (d + 1)(d + 2)/2 \leq 3(\log H)^2$. Since X is contained in $\#\Phi$ images of maps belonging to $M(B, C)$, by 2.4, $X(\mathbb{Q}, H)$ is contained in the union of at most

$$\#\Phi\ 63^C B(\log H)^{2C} H^{8/([\log H]+3)} \leq 63^C e^8 B\,\#\Phi\ (\log H)^{2C}$$

algebraic curves of degree d. Now applying 3.1 finds (if X has order r and degree (α, β))

$$N(X, H) \leq 63^C B e^8 \#\Phi (\log H)^{2C} 2^{r(r-1)/2+1} \log H \beta(\alpha + \log H\beta)^r$$

giving a bound of the required form (with $\gamma = 2C + r + 1$). □

4. Further remarks

4.1. EVIDENCE FOR 1.6. I consider the simplest examples where the functions in the chain are not bounded, namely the functions $\exp(-\frac{1}{x^n})$.

4.2. PROPOSITION. *Let n be a positive integer. Then the function*

$$\phi(x) = \exp\left(-\frac{1}{x^n}\right)$$

is mild.

Proof. Let, for $m > n$ and $x \in (0, 1)$

$$E(x) = E_{n,m}(x) = \exp\left(-\frac{1}{x^n}\right)\frac{1}{x^m}.$$

Then E' has a unique zero at $x = (n/m)^{1/n}$ which (checking $x \to 1$) gives the maximum of $E(x)$ on $(0, 1)$ as

$$\left(\frac{m}{n}\right)^{m/n} \exp(-m/n).$$

Now $\phi^{(k)}$ is a linear combination of such functions $E_{n,m}$. Write (for $k \geq 1$)

$$\phi^{(k)}(x) = \phi(x) \sum_{m=n+1}^{k(n+1)} a_m x^{-m} .$$

If $\alpha_k = \sum |a_m|$ then (differentiating with product rule) $\alpha_{k+1} \leq n\alpha_k + k(n+1)\alpha_k$. Thus $\alpha_{k+1} \leq (k+1)(n+1)\alpha_k$ and we conclude $\alpha_k \leq (n+1)^k k!$. Now maximizing the terms individually, the largest m/k is $k(n+1)/n$. So

$$\sup_{x\in(0,1)} \frac{|\phi^{(k)}(x)|}{k!} \leq \left(\frac{k(n+1)}{n}\right)^{k(n+1)/n} \exp\left(\frac{k(n+1)}{n}\right)(n+1)^k ,$$

having the required form with $C = \lambda = (n+1)/n$, $B = (e\lambda)^\lambda(n+1)$. □

4.3. FINAL REMARKS.

1. Conjecture 1.6 certainly holds for algebraic curves, which admit parameterization by a finite number of *analytic* functions. Moreover, by arguments of standard type (see, e.g., 5.1 and 5.2 in [12]) one obtains a uniform bound (for curves of given degree) for the number of parameterizing functions required, and the growth of their derivatives (i.e. the class $M(B,0)$ to which they belong; or alternately one may stipulate they belong to $M(1,0)$). This leads to the following further refinement of uniform bounds for the density function of an algebraic curve ([7, 10, 11]).

Let b, c be positive integers, and put $d = \max(b,c)$. There is a constant $\beta(d)$ with the following property. Suppose $P \in \mathbb{R}[x,y]$ is irreducible of bidegree (b,c). Let $X = \{(x,y) \in [-1,1]^2 : P(x,y) = 0\}$. Suppose $H \geq e^e$. Then

$$N(X,H) \leq \beta(d)\, H^{2/d} \log H .$$

2. It is natural to try to generalize the present results to higher dimensional pfaffian sets $X \subset [-1,1]^n$, on the assumption that they may be parameterized by images of $(0,1)^k, k < n$ with derivative bounds analogous to those considered here. An inductive process is then required as in [12]: first intersecting X with algebraic hypersurfaces, and then applying a lower-dimensional result to those intersections. Thus any such program will require a much more flexible result for curves than is obtained even assuming 1.6 to accommodate the implicitly defined sets that may arise.

Another requirement is a higher-dimensional analogue of 2.4 that is more precise than the formulation in [9; 4.4, 4.5], but this should not present special difficulties.

3. The method of this paper depends on two properties of a set $X \subset \mathbb{R}^2$. The first is good (uniform) control over the size of intersections $X \cap \Gamma$, where Γ is an algebraic curve of degree d. Call this control of *oscillation*. The second is a good (here *mild*) parameterization. Pfaffian functions enjoy the former. However, there are non-pfaffian functions which do as well [6]. It seems interesting to study whether there is a natural way to characterize individual functions having the necessary properties. Controlling oscillation only allows a subdivision and recurrence argument to substitute for a suitable parameterization; see [2, 9] and in particular [11] where this method is applied to pfaff curves, but yielding the weaker result stated in §1.

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